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SOLVING THE PROBLEM OF EXPANSION AND EXTENSION OF A HOLLOW CYLINDER WITH THE USE OF THE GRADIENT PLASTICITY THEORY

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An exact solution of the problem of expansion and extension of a hollow cylinder, which generalizes the known solutions based on the classical and gradient theories of plasticity under plane strain conditions, is obtained. The numerical solution reduces to calculating several ordinary integrals. The effect of the characteristic size of the material on the solution behavior near the cylinder orifice is studied.

Key words: *stress concentrator, characteristic size, gradient theory of plasticity.*

Gradient theories of plasticity are used to describe some effects arising near stress concentrators and during deformation of small-size bodies (see, e.g., [1–8]). In particular, Novopashin et al. [2] performed a detailed analytical analysis of gradient yield criteria and proposed their physical interpretation.

As with other theories, analytical and semi-analytical solutions are used to study the general properties of gradient models. In addition, exact solutions are used for testing numerical algorithms (see, e.g., [9]). Within the framework of the gradient theories of plasticity, solutions of this class were obtained for twisting a cylindrical rod [1] and expansion of a hollow cylinder under plane strain conditions [8]. One more exact solution is obtained in the present paper.

We use the gradient theory of plasticity proposed in [5], in which the yield condition has the form

$$\sigma_{\rm eq} = \sigma_0[f^2(\varepsilon_{\rm eq},\ \xi_{\rm eq}) + l_0|\nabla\varepsilon_{\rm eq}|]^{1/2}. \tag{1}$$

Here, $\sigma_{\rm eq} = \sqrt{3/2}\,(\tau_{ij}\tau_{ij})^{1/2}$ is the equivalent stress, τ_{ij} are the deviator components of the stress tensor, $\xi_{\rm eq} = \sqrt{2/3}\,(\xi_{ij}\xi_{ij})^{1/2}$ is the equivalent strain rate, ξ_{ij} are the components of the strain rate tensor, $\sigma_0 = \text{const}$, $\varepsilon_{\rm eq}$ is the equivalent plastic strain defined by the equation

$$\frac{d\varepsilon_{\rm eq}}{dt} = \xi_{\rm eq}, \tag{2}$$

and d/dt is the material derivative. The dependence of the function f included into Eq. (1) on the equivalent plastic strain and equivalent strain rate is usually taken in the same form as that used in the constitutive relations of the plasticity theory containing no gradient terms. In the present work, f is assumed to be independent of $\xi_{\rm eq}$, and the dependence of f on $\varepsilon_{\rm eq}$ is taken in the form of Swift's law [10]

$$f(\varepsilon_{\rm eq}) = (1 + \varepsilon_{\rm eq}/\varepsilon_0)^n, \qquad \varepsilon_0 = \text{const}, \quad n = \text{const}. \tag{3}$$

In Eq. (1), the gradient term is introduced with the use of the characteristic length l_0, which is a material property, and the gradient of the equivalent plastic strain $\nabla\varepsilon_{\rm eq}$. In addition to the plasticity condition (1), the constitutive equations include the associated plastic flow rule, which has the following form in the case considered:

$$\xi_{ij} = \lambda\tau_{ij}, \qquad \lambda \geqslant 0. \tag{4}$$

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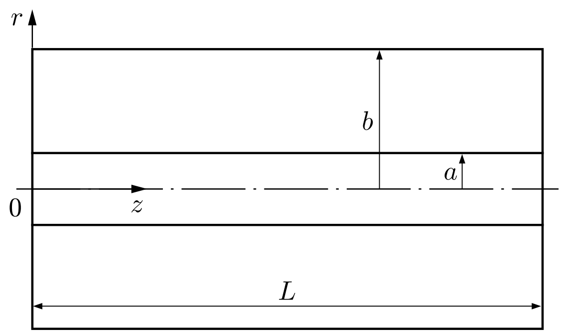

Fig. 1. Geometry of the problem.

Let us consider a hollow cylinder with the initial length L_0, initial inner radius a_0, and initial outer radius b_0. The current values of these quantities are denoted by L, a, and b, respectively. Let us introduce a cylindrical coordinate system $r\theta z$, where the z axis coincides with the axis of symmetry of the cylinder, and the end faces of the cylinder are described by the equations $z = 0$ and $z = L$ (Fig. 1). The stress–strain state is assumed to be independent of the z coordinate. In particular, the strain rate ξ_{zz} is determined in the form

$$\xi_{zz} = \xi_0 = L^{-1}\frac{dL}{dt}. \tag{5}$$

The outer surface of the cylinder is stress-free:

$$\sigma_{rr} = 0 \qquad \text{at} \quad r = b. \tag{6}$$

The time-independent radial velocity is set on the inner surface of the cylinder:

$$u = u_0 > 0 \qquad \text{at} \quad r = a. \tag{7}$$

Let us introduce the following dimensionless quantities:

$$\alpha = \frac{a}{a_0}, \quad q = \frac{a_0}{b_0}, \quad \rho = \frac{r}{b_0}, \quad l = \frac{l_0}{a_0}, \quad s = \frac{\xi_0 b_0}{2u_0}. \tag{8}$$

In the dimensionless variables (8), with allowance for axial symmetry and the fact that the stress–strain state is independent of z, the system of equations of the problem considered includes the equilibrium equation

$$\frac{\partial\sigma_{rr}}{\partial\rho} + \frac{\tau_{rr} - \tau_{\theta\theta}}{\rho} = 0 \tag{9}$$

and Eqs. (1)–(4). In particular, Eq. (4) yields the incompressibility equation reduced with the use of Eq. (5) to $\partial\,(\rho u)/\partial\rho = -\rho b_0\xi_0$. With allowance for Eq. (8), the general solution of this equation has the form

$$u/u_0 = s(\sqrt{3}\,\beta\rho^{-1} - \rho), \tag{10}$$

where β is an arbitrary function of time or α determined with the use of Eq. (7) as

$$\beta = (\alpha q/\sqrt{3}\,)(s^{-1} + \alpha q). \tag{11}$$

Taking into account Eqs. (10) and (11) and the incompressibility equation, we can present the quantities L and b in the form

$$L/L_0 = \exp\,[2sq(\alpha - 1)], \qquad (b/b_0)^2 = \alpha^2 q^2 + (1 - q^2)\exp\,[2sq(1 - \alpha)]. \tag{12}$$

The radial and circumferential strain rates are found from Eq. (10) as

$$\xi_{rr} = -\xi_0(1 + \sqrt{3}\,\beta\rho^{-2})/2, \qquad \xi_{\theta\theta} = \xi_0(\sqrt{3}\,\beta\rho^{-2} - 1)/2. \tag{13}$$

Substituting Eqs. (5) and (13) into the definition of the equivalent strain rate, we obtain

$$\xi_{\rm eq} = \xi_0\sqrt{1 + \beta^2\rho^{-4}}. \tag{14}$$

Using Eqs. (4), (5), (13), and (14), we can determine the deviator components of the stress tensor in the form

$$\tau_{zz} = \frac{2}{3}\,\frac{\rho^2\sigma_{\rm eq}}{(\rho^4+\beta^2)^{1/2}}, \qquad \tau_{\theta\theta} = -\frac{1}{3}\,\frac{(\rho^2-\sqrt{3}\,\beta)\sigma_{\rm eq}}{(\rho^4+\beta^2)^{1/2}}, \qquad \tau_{rr} = -\frac{1}{3}\,\frac{(\rho^2+\sqrt{3}\,\beta)\sigma_{\rm eq}}{(\rho^4+\beta^2)^{1/2}}. \tag{15}$$

Substituting these relations into the equilibrium equation (9), we find

$$\frac{\partial\sigma_{rr}}{\partial\rho} - \frac{2}{\sqrt{3}}\,\frac{\beta}{\rho}\,\frac{\sigma_{\rm eq}}{(\rho^4+\beta^2)^{1/2}} = 0. \tag{16}$$

To integrate Eq. (16), we have to determine the equivalent strain and its gradient involved into the expression for $\sigma_{\rm eq}$.

It follows from Eq. (7) that $da/dt = u_0$. Then, taking into account that $\varepsilon_{\rm eq}$ does not depend on z and using the dimensionless quantities (8), we apply Eqs. (2), (10), and (14) to obtain

$$\frac{\partial\varepsilon_{\rm eq}}{\partial\alpha} + sq(\sqrt{3}\beta\rho^{-1}-\rho)\,\frac{\partial\varepsilon_{\rm eq}}{\partial\rho} = 2sq(1+\beta^2\rho^{-4})^{1/2}. \tag{17}$$

The characteristics of Eq. (17) are determined by the equation

$$\frac{d\rho}{d\alpha} = sq(\sqrt{3}\,\beta\rho^{-1}-\rho). \tag{18}$$

With allowance for Eq. (11), the general solution of Eq. (18) can be written as

$$\rho^2 = q^2\alpha^2 + (\rho_0^2 - q^2)\exp\,[2sq(1-\alpha)], \tag{19}$$

where ρ_0 is the constant of integration. Obviously, we have $\rho = \rho_0$ at the beginning of the deformation process (at $\alpha = 1$). In particular, we have $\rho = b/b_0$ at $\rho_0 = 1$, and Eq. (19) yields the dependence of b/b_0 on α, which coincides with Eq. (12). To simplify the form of the equations, we introduce the function of α and ρ_0 in the form

$$\Lambda(\alpha,\rho_0) = q^2\alpha^2 + (\rho_0^2 - q^2)\exp\,[2sq(1-\alpha)]. \tag{20}$$

Taking into account Eqs. (11), (19), and (20), we use Eq. (17) to obtain the relation along the characteristics in the form

$$\frac{d\varepsilon_{\rm eq}}{d\alpha} = 2sq\Big(1+\frac{q^2\alpha^2(s^{-1}+\alpha q)^2}{3\Lambda(\alpha,\rho_0)^2}\Big)^{1/2}, \tag{21}$$

where ρ_0 is assumed to be constant on each characteristic. As we have $\varepsilon_{\rm eq} = 0$ at $\alpha = 1$, then the solution of Eq. (21) is

$$\varepsilon_{\rm eq} = 2sq\int\limits_1^\alpha \Big(1+\frac{q^2\alpha^2(s^{-1}+\alpha q)^2}{3\Lambda(\alpha,\rho_0)^2}\Big)^{1/2}\,d\alpha. \tag{22}$$

It is convenient to use the following independent variables instead of α and ρ:

$$\alpha' = \alpha, \qquad \rho_0 = \{q^2 + (\rho^2 - q^2\alpha^2)\exp\,[2sq(\alpha-1)]\}^{1/2}. \tag{23}$$

The expression for ρ_0 is derived from Eq. (19). Taking into account Eq. (23), we find the dependence between the derivatives in the new and old variables:

$$\frac{\partial}{\partial\rho} = \frac{\partial}{\partial\rho_0}\,\frac{\partial\rho_0}{\partial\rho}. \tag{24}$$

The derivative $\partial\rho_0/\partial\rho$ is found from Eq. (23) with the use of Eqs. (19) and (20):

$$\frac{\partial\rho_0}{\partial\rho} = \Lambda(\alpha,\rho_0)^{1/2}\exp\,[2sq(\alpha-1)]\rho_0^{-1}. \tag{25}$$

In the new variables, the derivative $\partial\varepsilon_{\rm eq}/\partial\rho_0$ can be calculated from Eq. (22) under the assumption that $\alpha = \alpha' = \text{const}$. Then, we obtain

$$\frac{\partial\varepsilon_{\rm eq}}{\partial\rho_0} = -\frac{4sq^3\rho_0}{\sqrt{3}}\int\limits_1^\alpha \frac{\alpha^2(s^{-1}+\alpha q)^2\exp\,[2sq(1-\alpha)]}{[\alpha^2q^2(s^{-1}+\alpha q)^2 + 3\Lambda(\alpha,\rho_0)^2]^{1/2}\Lambda(\alpha,\rho_0)^2}. \tag{26}$$

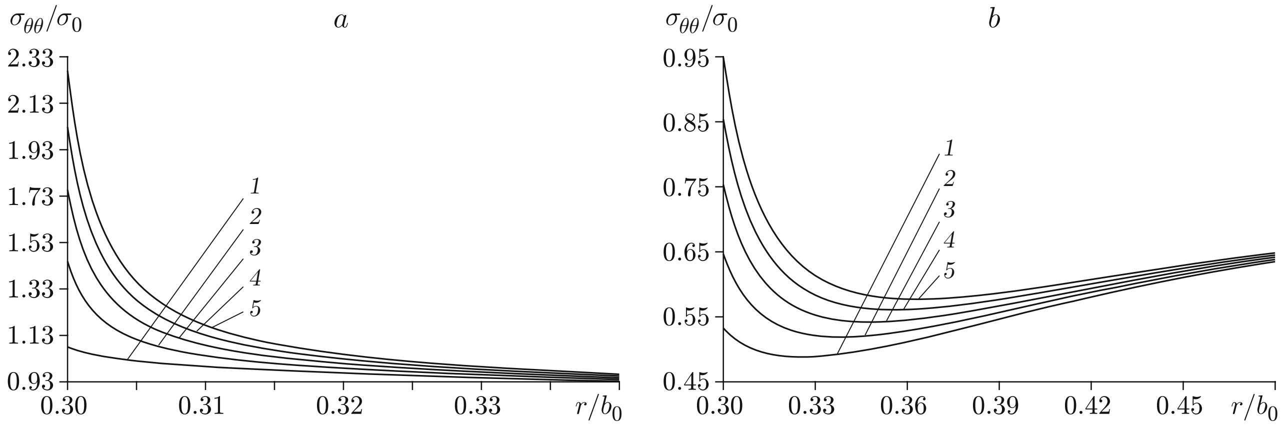

Fig. 2. Circumferential stress versus the radius near the stress concentrator for $s = 5$ (a) and 0.5 (b): $l = 0$ (no gradient terms) (1), 0.1 (2), 0.2 (3), 0.3 (4), and 0.4 (5).

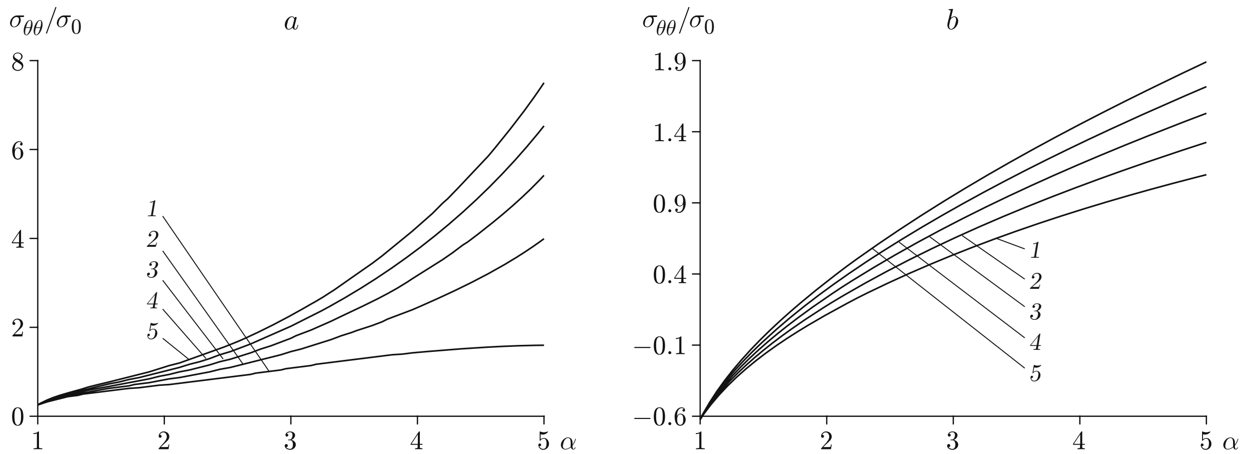

Fig. 3. Circumferential stress on the inner surface of the cylinder versus α (notation the same as in Fig. 2).

Substituting Eqs. (25) and (26) into Eq. (24), we obtain

$$\frac{\partial \varepsilon_{\rm eq}}{\partial \rho} = -\frac{4sq^3}{\sqrt{3}}\,\Lambda(\alpha,\rho_0)^{1/2}\exp\,[2sq(\alpha-1)]\int\limits_1^{\alpha}\frac{\alpha^2(s^{-1}+\alpha q)^2\exp\,[2sq(1-\alpha)]}{[\alpha^2q^2(s^{-1}+\alpha q)^2+3\Lambda(\alpha,\rho_0)^2]^{1/2}\Lambda(\alpha,\rho_0)^2}. \qquad (27)$$

In the case considered, we have $|\nabla\varepsilon_{\rm eq}| = |\partial\varepsilon_{\rm eq}/\partial\rho|b_0^{-1}$; therefore, Eqs. (22) and (27) together with Eqs. (1) and (3) allow us to determine the quantity $\sigma_{\rm eq}$, which is involved into Eq. (16), as a function of α and ρ_0. To integrate Eq. (16), it is convenient to pass to differentiation with respect to ρ_0 with the use of Eqs. (24) and (25). As a result, we obtain

$$\frac{\partial\sigma_{rr}}{\partial\rho_0} = \frac{2\alpha q(s^{-1}+\alpha q)\rho_0\sigma_{\rm eq}}{\sqrt{3}\,\Lambda(\alpha,\rho_0)\exp\,[2sq(\alpha-1)][3\Lambda(\alpha,\rho_0)^2+(s^{-1}+\alpha q)^2\alpha^2q^2]^{1/2}}.$$

Eliminating $\sigma_{\rm eq}$ from the last equation with the help of Eqs. (1) and (3), we write its solution satisfying condition (6) as

$$\frac{\sigma_{rr}}{\sigma_0} = \frac{2\alpha q(s^{-1}+\alpha q)}{\sqrt{3}\,\exp\,[2sq(\alpha-1)]}\int\limits_1^{\rho_0}\frac{\rho_0[(1+\varepsilon_{\rm eq}/\varepsilon_0)^{2n}+ql|\partial\varepsilon_{\rm eq}/\partial\rho|]^{1/2}}{\Lambda(\alpha,\rho_0)[3\Lambda(\alpha,\rho_0)^2+(s^{-1}+\alpha q)^2\alpha^2q^2]^{1/2}}\,d\rho_0. \qquad (28)$$

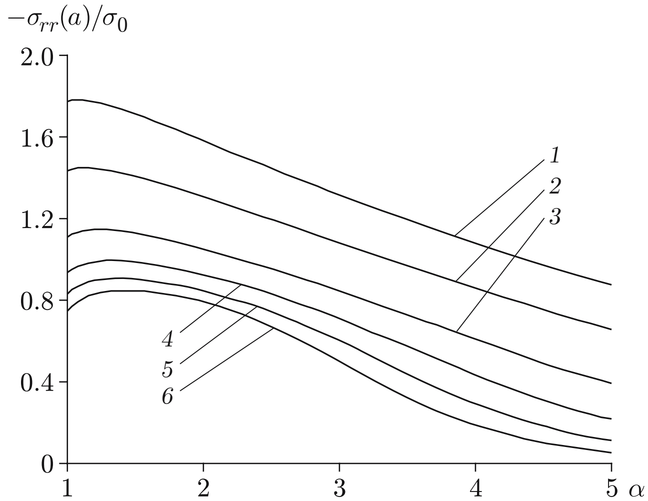

Fig. 4. Radial stress on the inner surface of the cylinder versus α: $s = 0.5$, 1 (2), 2 (3), 3 (4), 4 (5), and 5 (6).

The stress that should be applied to the surface $r = a$ is found from this relation with ρ_0 being replaced by q in the upper limit of the integral. Using Eqs. (22) and (27), we can perform integration in Eq. (28) numerically, after which the circumferential stress is determined with the help of Eq. (15) by the formula

$$\frac{\sigma_{\theta\theta}}{\sigma_0} = \frac{\sigma_{rr}}{\sigma_0} + \frac{\tau_{\theta\theta} - \tau_{rr}}{\sigma_0} = \frac{\sigma_{rr}}{\sigma_0} + \frac{2}{\sqrt{3}} \frac{\alpha q(s^{-1} + \alpha q)[(1 + \varepsilon_{\text{eq}}/\varepsilon_0)^{2n} + ql|\partial\varepsilon_{\text{eq}}/\partial\rho|]^{1/2}}{[3\Lambda(\alpha, \rho_0)^2 + \alpha^2 q^2 (s^{-1} + \alpha q)^2]^{1/2}}. \qquad (29)$$

For a number of aluminum alloys, we can assume that $\varepsilon_0 = 0.222$ and $n = 0.25$ [11]; hence, these values and also the value $q = 0.1$ are used in all cases considered below. Of greatest interest is the dependence of the solution near the stress concentrator on l. In the parametric form, the dependence of $\sigma_{\theta\theta}$ on r is determined from Eqs. (19) and (29). Figure 2 shows the behavior of $\sigma_{\theta\theta}(r)$ in the vicinity of the surface $r = a$ at $\alpha = 3$ and different values of l. The radial distribution of the stress σ_{rr} is almost independent of l. The circumferential stress is plotted in Fig. 3 as a function of α at $r = a$ and different values of l. At $r = a$ and a fixed value of α, the radial stress is almost independent of l. For the deformation process to occur, this pressure should be applied to the inner surface. The dependence of $\sigma_{rr}(a)$ on α at $l = 0$ is plotted in Fig. 4. It follows from the results in Figs. 2–4 that the characteristic size of the material exerts a significant effect on the circumferential stress distribution in the vicinity of the stress concentrator.

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